Estimation and Filtering of Gaussian Variables with Linear Inequality Constraints

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Abstract— **In practice, a parameter or the state of a system is often subject to constraints. This paper considers the estimation problem for the parameter or the state constrained by a class of linear inequalities. Two sequential methods for optimal parameter estimation in the MMSE sense are obtained. They have an analytic form, which is different from most existing methods. For a dynamic system with constrained state, we model it with density function and provide a suboptimal filter based on reasonable approximations. This filter is applied to an example of tracking a ground moving target and its performance is also examined.**

Keywords: Inequality constraint, optimal estimation, filtering.

I. INTRODUCTION

In many practical applications, parameters or state variables are often subject to constraints. These constraints commonly contain valuable information that should be taken into account in the estimation process, and theoretically this prior knowledge of the constraints can improve estimation accuracy. However, incorporating the constraints sometimes makes the modeling difficult. Thus, constraints are often neglected in estimation problems.

Due to their effects on the estimation performance improvement, constraints have attracted much research attention in recent years. For the case that the state variables are subject to equality constraints, one method is to treat these constraints as perfect (or noise-free) measurements [1]. But it has some disadvantages in a Kalman filter implementation, as mentioned in [2]. For example, the singular error covariance of the perfect measurements may induce numerical problems and the augmented measurements increase the computational load. To avoid numerical problems, one heuristic way is to replace the perfect measurements by one with small noise [3] (then the hard constraints become soft ones), but the optimality is lost. In [4], the state estimation with equality constraints is formulated as the problem with noisy and noise-free measurements, and the best linear unbiased estimate is obtained. Another popular approach for equality constrained estimation is the projection method [2] [5], in which the unconstrained estimate is projected onto the constraint subspace. A variety of projection methods is proposed in [6] to restrict the optimal Kalman gain so that the updated state estimate will not violate the constraints.

Compared with the state estimation with equality constraints, it is more general in applications that the state, or part of the state is constrained by bounds, such as maximum amplitude of a signal and maximum attainable speed of a motor. This prior information of the inequality constraints should be made full use of and some research efforts have been made in this field. [7] considers a general linear model when the parameter space is subject to linear inequality constraints and presents a Bayesian analysis of this model. [8] provides an approach using the Gibbs sampling for Bayesian inference subject to inequality constraints. Under the white Gaussian noise assumption, [9] gives an analytic expression of minimum variance estimation for one entry of the parameter vector constrained by bounds. The methods of [10], [11], [6], [12] incorporate state inequality constraints into the Kalman filter and the resultant filter is a combination of the Kalman filter and the solution of a quadratic programming problem. Other strategies based on the Monte Carlo method are discussed in [13] [14]. In addition, moving horizon estimation is a robust approach for constrained state estimation, and a sufficient condition for its stability is derived in [15].

This paper considers the problem of estimating parameters constrained by a class of linear inequalities. We model the constrained parameters to have a truncated Gaussian distribution. In the Bayesian framework, two sequential methods for the optimal parameter estimation in the MMSE sense are obtained. They have an analytic form which is different from most existing methods.

For a dynamic system with constrained state, we model it with density function and provide a suboptimal filter based on reasonable approximations. This filter is applied to an example of tracking a ground moving target and its performance is examined.

This paper is organized as follows. The first two moments of a truncated Gaussian random variable are calculated in Section II. In Section III, an optimal estimator for a parameter with linear inequality constraints is obtained. Section VI develops

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a suboptimal filter for the dynamic system with constrained state, and its application for the example of tracking a ground moving target is presented in Section V. The last section presents the concluding remarks and future research.

II. Mean and Covariance of Truncated Gaussian Random Variable

Let $\mathcal{N}(\mathbf{x};\bar{\mathbf{x}},\mathbf{P})$ denote the Gaussian probability density function (pdf)

$$p\left(\mathbf{x}\right)=\left|2\pi\mathbf{P}\right|^{-1/2}\exp\left[-\frac{1}{2}\left(\mathbf{x}-\bar{\mathbf{x}}\right)'\mathbf{P}^{-1}\left(\mathbf{x}-\bar{\mathbf{x}}\right)\right] \quad (1)$$

where $\bar{\mathbf{x}}$ and $\mathbf{P}$ are the mean and covariance of the random variable $\mathbf{x}$. If the Gaussian variable $\mathbf{x}$ is constrained by domain $\mathcal{A}$, then the conditional pdf of $\mathbf{x}$, referred to as truncated Gaussian, is

$$p\left(\mathbf{x}|\mathbf{x}\in\mathcal{A}\right)=\frac{1}{c}\mathcal{N}(\mathbf{x};\bar{\mathbf{x}},\mathbf{P})\mathbf{1}_{\{\mathbf{x}\in\mathcal{A}\}} \quad (2)$$

where $\mathbf{1}_{\{\cdot\}}$ is the indicator function and c is the normalization factor.

We consider the case that $\mathbf{x}$ is restricted by a linear inequality

$$\mathcal{A}=\{\mathbf{x}:a\leq\alpha'\mathbf{x}\leq b\} \quad (3)$$

where α is a vector whose dimension equals that of $\mathbf{x}$, and a and b are both scalars. To obtain the conditional expectation of $\mathbf{x}$ under the linear inequality constraint $\mathbf{x}\in\mathcal{A}$, we have the following proposition.

Proposition 1: Assume that the distribution of the random variable $\mathbf{x}$ is Gaussian with mean $\bar{\mathbf{x}}$ and covariance $\mathbf{P}$, namely, $\mathbf{x}\sim\mathcal{N}(\bar{\mathbf{x}},\mathbf{P})$. If $\mathbf{x}$ is constrained by $\mathcal{A}$ of (3), then the conditional expectation of $\mathbf{x}$ and its corresponding covariance are

$$E\left(\mathbf{x}|\mathbf{x}\in\mathcal{A}\right)=\bar{\mathbf{x}}+K\left(m-\alpha'\bar{\mathbf{x}}\right) \quad (4)$$

$$\mathrm{Cov}\left(\mathbf{x}|\mathbf{x}\in\mathcal{A}\right)=\mathbf{P}-K\left(\alpha'\mathbf{P}\alpha-\Sigma\right)K' \quad (5)$$

where

$$K=\mathbf{P}\alpha\left(\alpha'\mathbf{P}\alpha\right)^{-1}$$

$$m=-\frac{\alpha'\mathbf{P}\alpha}{c}[\mathcal{N}(b;\alpha'\bar{\mathbf{x}},\alpha'\mathbf{P}\alpha)-\mathcal{N}(a;\alpha'\bar{\mathbf{x}},\alpha'\mathbf{P}\alpha)]+\alpha'\bar{\mathbf{x}}$$

$$\Sigma=-\frac{\alpha'\mathbf{P}\alpha}{c}[(b+\alpha'\bar{\mathbf{x}})\mathcal{N}(b;\alpha'\bar{\mathbf{x}},\alpha'\mathbf{P}\alpha) -(a+\alpha'\bar{\mathbf{x}})\mathcal{N}(a;\alpha'\bar{\mathbf{x}},\alpha'\mathbf{P}\alpha)]+\alpha'\bar{\mathbf{x}}\bar{\mathbf{x}}'\alpha+\alpha'\mathbf{P}\alpha-m^2$$

and the normalization factor

$$c=\int_a^b\mathcal{N}(x;\alpha'\bar{\mathbf{x}},\alpha'\mathbf{P}\alpha)\mathrm{d}x$$

A proof of Proposition 1 is provided in the Appendix.

Remark: a) Compared with the solution by numerical method [8] or by optimization [12], this estimate of the variable with linear inequality constraints is analytic and its computational load is low. This proposition was derived from the estimation with quantized measurements whose error variance was set to zero in [16] and [17]. b) The minimum variance estimation of a parameter constrained by bounds presented in [9] can be regarded as a special case of our linear inequality constraints with $\alpha=[1\ 0\ \cdots\ 0]'$.

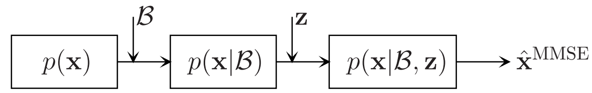

Fig. 1. Method A

III. Optimal Estimation for a Parameter with Linear Inequality Constraints

Assume that the prior pdf of static parameter $\mathbf{x}$ is Gaussian, namely, $p\left(\mathbf{x}\right)=\mathcal{N}(\mathbf{x};\bar{\mathbf{x}},\mathbf{P})$, and linear inequality constraints are placed on the estimand $\mathbf{x}$, defined by (3). The measurement model is defined as

$$\mathbf{z}=H\mathbf{x}+v \quad (6)$$

where v is uncorrelated with $\mathbf{x}$ and $E\left(v\right)=0$ and $E\left(v_kv_j'\right)=R\delta_{kj}$. From the measurement model (6), we can easily see that it satisfies the necessary and sufficient condition for the *linear* MMSE (LMMSE) estimator to be recursible [18]. The LMMSE estimator is actually the MMSE estimator if $(\mathbf{x},\mathbf{z})$ is Gaussian, so the MMSE estimator in this case has a recursive form. In this section, $\mathbf{z}=\{\mathbf{z}_1,\cdots,\mathbf{z}_k\}$ stands for a data (set). It can be either an individual measurement or a stacked one, since here, as analyzed above, the recursive MMSE estimation has the same performance as the batch MMSE estimation. Also, let the measurement matrix be H and the covariance of the Gaussian measurement noise v be R.

Similar to [1], [3], we treat the constraints as a pseudo-measurement constrained by bounds, that is,

$$\mathcal{B}=\{\mathbf{x}\in\mathcal{A}\} =\{z_c=\alpha'\mathbf{x}\text{ and }z_c\in[a,b]\}$$

where vector α and scalars a and b are all known. This section is to obtain the MMSE estimate of $\mathbf{x}$ given the constraints and a data set $\mathbf{z}$.

A. Constrained MMSE Estimation

As is known, conditional mean $E\left(\mathbf{x}|z\right)$ minimizes conditional (and unconditional) mean squared error (MSE) matrix (or scalar) among all estimators [19]. Note that z is arbitrary. It may be the measurement $\mathbf{z}$ and the bounded pseudo-measurement z_c. Therefore,

$$\hat{\mathbf{x}}^{\mathrm{MMSE}}=E\left(\mathbf{x}|\mathbf{z},\mathcal{B}\right)=\int\mathbf{x}p\left(\mathbf{x}|\mathbf{z},\mathcal{B}\right)\mathrm{d}\mathbf{x} \quad (7)$$

In order to obtain the posterior pdf $p\left(\mathbf{x}|\mathbf{z},\mathcal{B}\right)$, we adopt the following two different methods.

1) Method A

As illustrated in Fig. 1, method A is to obtain $p\left(\mathbf{x}|\mathcal{B}\right)$ first and then obtain $p\left(\mathbf{x}|\mathbf{z},\mathcal{B}\right)$. Next, we give its details.

Applying Bayes' rule, we have

$$p\left(\mathbf{x}|\mathcal{B}\right)\propto p\left(\mathbf{x}\right)\mathbf{1}_{\{\mathbf{x}\in\mathcal{A}\}}=\mathcal{N}(\mathbf{x};\bar{\mathbf{x}},\mathbf{P})\mathbf{1}_{\{\mathbf{x}\in\mathcal{A}\}}$$

where $\propto$ stands for "proportional to". Then, applying Bayes' rule again, we have

$$p\left(\mathbf{x}|\mathbf{z},\mathcal{B}\right)\propto p\left(\mathbf{z}|\mathbf{x}\right)p\left(\mathbf{x}|\mathcal{B}\right) \propto\mathcal{N}(\mathbf{z};H\mathbf{x},R)\mathcal{N}(\mathbf{x};\bar{\mathbf{x}},\mathbf{P})\mathbf{1}_{\{\mathbf{x}\in\mathcal{A}\}} \quad (8)$$

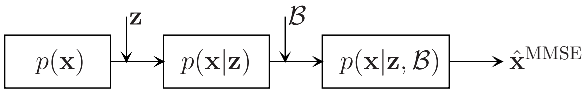

Fig. 2. Method B

The product of the two Gaussian pdfs in (8) is the update step of the Kalman filter with the linear-Gaussian assumption, and its result is the updated Gaussian density of $\mathbf{x}$ [20], that is,

$$\mathcal{N}(\mathbf{z}; H\mathbf{x}, R)\mathcal{N}(\mathbf{x}; \bar{\mathbf{x}}, \mathbf{P}) \propto \mathcal{N}(\mathbf{x}; \breve{\mathbf{x}}, \breve{\mathbf{P}}) \quad (9)$$

where

$$\breve{\mathbf{x}} = \bar{\mathbf{x}} + K(\mathbf{z} - H\bar{\mathbf{x}}) \quad (10)$$
$$\breve{\mathbf{P}} = \mathbf{P} - KSK' \quad (11)$$

and

$$S = H\mathbf{P}H' + R$$
$$K = \mathbf{P}H'S^{-1}$$

Eq. (9) is important to bridge the Kalman filter and Bayesian estimation. A proof of it can be found in [21].

From (8) and (9), we have

$$p(\mathbf{x}|\mathcal{B}, \mathbf{z}) \propto \mathcal{N}(\mathbf{x}; \breve{\mathbf{x}}, \breve{\mathbf{P}})\mathbf{1}_{\{\mathbf{x}\in\mathcal{A}\}}$$

2) Method B

Fig. 2 illustrates another way to obtain the posterior pdf $p(\mathbf{x}|\mathbf{z},\mathcal{B})$: First, obtain $p(\mathbf{x}|\mathbf{z})$ given the measurement $\mathbf{z}$, and then use the constraints $\mathcal{B}$ as a bounded pesudo-measurement z_c to obtain $p(\mathbf{x}|\mathbf{z},\mathcal{B})$. From the analysis above for method A, it is easily shown that $p(\mathbf{x}|\mathbf{z}) = \mathcal{N}(\mathbf{x}; \breve{\mathbf{x}}, \breve{\mathbf{P}})$ with $\breve{\mathbf{x}}$ and $\breve{\mathbf{P}}$ given by (10) and (11). The posterior density $p(\mathbf{x}|\mathbf{z})$ is truncated by the linear inequality constraints $\mathcal{B}$, so

$$p(\mathbf{x}|\mathbf{z},\mathcal{B}) \propto \mathcal{N}(\mathbf{x}; \breve{\mathbf{x}}, \breve{\mathbf{P}})\mathbf{1}_{\{\mathbf{x}\in\mathcal{A}\}}$$

Note that these two methods lead to the same result. Since $p(\mathbf{x}|\mathbf{z},\mathcal{A})$ is a probability density which integrates to 1, we have

$$p(\mathbf{x}|\mathbf{z},\mathcal{B}) = \frac{1}{c}\mathcal{N}(\mathbf{x}; \breve{\mathbf{x}}, \breve{\mathbf{P}})\mathbf{1}_{\{\mathbf{x}\in\mathcal{A}\}} \quad (12)$$

where the normalization factor $c = \int_a^b \mathcal{N}(z_c; \alpha'\breve{\mathbf{x}}, \alpha'\breve{\mathbf{P}}\alpha)\mathrm{d}z_c$. Then, the MMSE estimate of $\mathbf{x}$ can be derived from the standard solution in Proposition 1.

B. Properties of the Constrained Estimation

The previous section gives the results of MMSE-optimal estimation for the parameters with linear inequality constraints. In this section, we discuss their properties. Let $\breve{\mathbf{x}}$, $\breve{\mathbf{P}}$ be the estimate (i.e., the expectation and MSE) of the parameter without constraints and $\hat{\mathbf{x}}^*$, $\mathbf{P}^*$ the estimate with linear inequality constraints.

PROPERTY 1: The constrained estimate $\hat{\mathbf{x}}^*$ is unbiased, that is,

$$E(\hat{\mathbf{x}}^*) = E(\mathbf{x})$$

Proof: From Equation (4), we have

$$\mathbf{x} - \hat{\mathbf{x}}^* = \mathbf{x} - \breve{\mathbf{x}} - K[\alpha' E(\mathbf{x}|\mathcal{B}) - \alpha'\breve{\mathbf{x}}] \quad (13)$$

Taking expectation on both sides of (13) and by the total expectation theorem, we have

$$E[E(\mathbf{x}|\mathcal{B})] = E(\mathbf{x}) \quad (14)$$

Thus,

$$E(\mathbf{x} - \hat{\mathbf{x}}^*) = (I - K\alpha') E(\mathbf{x} - \breve{\mathbf{x}}) \quad (15)$$

Since $\breve{\mathbf{x}}$ is the MMSE estimate of the parameter without constraints, $\breve{\mathbf{x}}$ is unbiased, i.e., $E(\breve{\mathbf{x}}) = E(\mathbf{x})$. Therefore the right hand of (15) is zero, which implies that the estimate of $\mathbf{x}$ with inequality constraints is unbiased. ■

PROPERTY 2: The inequality constrained estimate $\hat{\mathbf{x}}^*$ has a smaller error covariance than the unconstrained estimate $\breve{\mathbf{x}}$. That is,

$$\mathrm{Cov}(\hat{\mathbf{x}}^*) < \mathrm{Cov}(\breve{\mathbf{x}}) \quad (16)$$

Here, the notation $A < B$ indicates $A - B$ is negative definite.

Remark: A similar result for the state estimate with equality constraints has been obtained in [2]. For the case with inequality constraints, (16) can be proved through routine calculation. Here, we provide an intuitive justification: Constraints reduce the uncertainty of the estimand (i.e., the quantity to be estimated), which means that the optimal constrained estimation must also have reduced uncertainty. That is, the covariance of $\hat{\mathbf{x}}^*$ is smaller than that of $\breve{\mathbf{x}}$.

PROPERTY 3: Under the assumptions that variable $\mathbf{x}$ is with Gaussian pdf and is observed by a stochastic process $\mathbf{z}$ with additive Gaussian white noise, the MMSE-optimal estimate $\hat{\mathbf{x}}^* = E[\mathbf{x}|\mathbf{z}_1, \mathbf{z}_2, \mathcal{B}]$ is recursible. Differing from the standard recursive form described in [18], the recursive form of $\hat{\mathbf{x}}^*$ needs the following steps with the known initial pdf $p(\mathbf{x})$ of $\mathbf{x}$, and the processing can be generalized to the total measurement sequence.

- Step 1: By Bayes' rule, obtain the posterior pdf $p(\mathbf{x}|\mathbf{z}_1)$, and then impose the constraints $\mathcal{B}$ to $p(\mathbf{x}|\mathbf{z}_1)$ and get $p(\mathbf{x}|\mathbf{z}_1, \mathcal{B})$; based on Proposition 1, produce the MMSE estimate of $\mathbf{x}$ conditioned on $\mathbf{z}_1$.
- Step 2: Obtain $p(\mathbf{x}|\mathbf{z}_1, \mathbf{z}_2)$ based on $p(\mathbf{x}|\mathbf{z}_1)$ and $\mathbf{z}_2$, and then impose the constraints $\mathcal{B}$ to $p(\mathbf{x}|\mathbf{z}_1, \mathbf{z}_2)$ and get $p(\mathbf{x}|\mathbf{z}_1, \mathbf{z}_2, \mathcal{B})$; using the result in Proposition 1, produce the MMSE estimate of $\mathbf{x}$ conditioned on $\mathbf{z}_1$ and $\mathbf{z}_2$.

IV. FILTERING FOR THE LINEARLY CONSTRAINED STATE

In this section, we discuss the filtering problem for a discrete-time linear system with its state constrained by the following linear inequality:

$$a \le \alpha'\mathbf{x}_k \le b, \quad k = 0, 1, \cdots \quad (17)$$

where a, b and α were defined in (3). The question is often solved by the *Kalman filter with constraints* [10] [6]. The main idea can be described briefly as follows. First, model the linear system as

$$\mathbf{x}_{k+1} = F_k \mathbf{x}_k + w_k \tag{18}$$
$$\mathbf{z}_k = H_k \mathbf{x}_k + v_k \tag{19}$$

with the standard conditions:

$$E\left(w_k w_j'\right) = Q_k \delta_{kj} \tag{20}$$
$$E\left(v_k v_j'\right) = R_k \delta_{kj} \tag{21}$$
$$E\left(v_k w_j'\right) = \mathbf{0},\ E\left(\mathbf{x}_k v_j'\right) = \mathbf{0} \tag{22}$$
$$E\left(\mathbf{x}_k w_j'\right) = \mathbf{0} \quad (j \geq k) \tag{23}$$

The optimal estimation for the unconstrained state in the LMMSE (MMSE if the noises are Gaussian) sense can be obtained by the Kalman filter. Then, project the solution onto the constrained surface and the constrained state estimate is obtained. This procedure is analogous to that of the constrained parameter estimate in the previous section.

The primary disadvantage of this method (roughly called projection method) lies in the oversight of the correlation in modeling the system with state constraints; that is, the constraints make (23) invalid, which is a necessary conditions for the Kalman filter to be optimal.

As discussed, the inequality constraints cannot be directly incorporated into a Kalman filter algorithm without severe approximations, but it would be better to make approximations as late as possible, not to the system model. Here, we model the system with density function:

$$p\left(\mathbf{x}_{k+1}|\mathbf{x}_k\right) = \frac{1}{c}\mathcal{N}(\mathbf{x}_{k+1}; F_k\mathbf{x}_k, Q_k)\mathbf{1}_{\{\alpha'\mathbf{x}_{k+1}\in[a,b]\}} \tag{24}$$
$$p\left(\mathbf{z}_k|\mathbf{x}_k\right) = \mathcal{N}(\mathbf{z}_k; H_k\mathbf{x}_k, R_k), \qquad k = 0, 1, \cdots \tag{25}$$

Assume the initial state $\mathbf{x}_0 \sim \mathcal{N}(\hat{\mathbf{x}}_0, \mathbf{P}_0)$, and a measurement sequence $Z^k = \{\mathbf{z}_1, \cdots, \mathbf{z}_k\}$ is available. The goal is to obtain the posterior density $p\left(\mathbf{x}_k|\mathbf{x}_0, Z^k\right)$ and it can be calculated successively as follows.
The predicted pdf of $\mathbf{x}_1$:

$$\begin{aligned} p\left(\mathbf{x}_1\right) &= \int p\left(\mathbf{x}_1|\mathbf{x}_0\right) p\left(\mathbf{x}_0\right) \mathrm{d}\mathbf{x}_0 && (26) \\ &\propto \int \mathcal{N}(\mathbf{x}_1; F_0\hat{\mathbf{x}}_0, Q_0)\mathbf{1}_{\{\alpha'\mathbf{x}_1\in[a,b]\}}\mathcal{N}(\hat{\mathbf{x}}_0, \mathbf{P}_0)\mathrm{d}\mathbf{x}_0 \\ &\propto \mathcal{N}(\mathbf{x}_1; F_0\hat{\mathbf{x}}_0, F_0\mathbf{P}_0F_0' + Q_0)\mathbf{1}_{\{\alpha'\mathbf{x}_1\in[a,b]\}} \\ &\triangleq \mathcal{N}(\mathbf{x}_1; \hat{\mathbf{x}}_{1|0}, \mathbf{P}_{1|0})\mathbf{1}_{\{\alpha'\mathbf{x}_1\in[a,b]\}} && (27) \end{aligned}$$

The updated pdf of $\mathbf{x}_1$:

$$\begin{aligned} p\left(\mathbf{x}_1|\mathbf{z}_1\right) &\propto p\left(\mathbf{z}_1|\mathbf{x}_1\right) p(\mathbf{x}_1) \\ &\propto \mathcal{N}(\mathbf{x}_1; \hat{\mathbf{x}}_1, \mathbf{P}_1)\mathbf{1}_{\{\alpha'\mathbf{x}_1\in[a,b]\}} && (28) \end{aligned}$$

where

$$\hat{\mathbf{x}}_1 = \hat{\mathbf{x}}_{1|0} + \mathbf{K}\left(\mathbf{z}_1 - H_1\hat{\mathbf{x}}_{1|0}\right)$$
$$\mathbf{P}_1 = \mathbf{P}_{1|0} - \mathbf{K}\mathbf{S}\mathbf{K}'$$

and

$$\mathbf{S} = H_1\mathbf{P}_{1|0}H_1' + R_1$$
$$\mathbf{K} = \mathbf{P}_{1|0}H'\mathbf{S}^{-1}$$

Now based on Proposition 1, we can derive the analytic form of the mean $\hat{\mathbf{x}}_1^*$ and covariance $\mathbf{P}_1^*$ of the constrained state $\mathbf{x}_1$ conditioned on $\mathbf{z}_1$.

Therefore, under the assumption that the initial state is Gaussian, the MMSE-optimal estimate of $\mathbf{x}_1$ conditioned on $\mathbf{z}_1$ can be obtained. Unfortunately, there is no analytic form for the integral (26) in the next recursion, and $p(\mathbf{x}_2|\mathbf{x}_1, \mathbf{z}_1)$ is no longer a truncated Gaussian. To overcome the difficulty in evaluating the integral, we make a reasonable approximation, although the integral can be computed by Monte Carlo integration and theoretically its value can be obtained exactly only if the sample size is large enough. We first introduce another Proposition [22].

Proposition 2: Let $p(\mathbf{x}; \bar{\mathbf{x}}^*, \mathbf{P}^*)$ denote the pdf $p\left(\mathbf{x}\right)$ (not necessarily Gaussian) of $\mathbf{x}$ with mean $\bar{\mathbf{x}}^*$ and covariance $\mathbf{P}^*$. The problem is to approximate the pdf of $\mathbf{x}$ with the Gaussian pdf $g\left(\mathbf{x}\right) = \mathcal{N}(\mathbf{x}; \bar{\mathbf{x}}, \mathbf{P})$. It turns out that $(\bar{\mathbf{x}}, \mathbf{P}) = (\bar{\mathbf{x}}^*, \mathbf{P}^*)$ minimizes the Kullback-Leibler (KL) discrimination $D_{\text{KL}}\left(p\left(\mathbf{x}\right), g\left(\mathbf{x}\right)\right)$:

$$(\bar{\mathbf{x}}^*, \mathbf{P}^*) = \arg\min_{[\bar{\mathbf{x}}, \mathbf{P}]} D_{\text{KL}}\left(p(\mathbf{x}; \bar{\mathbf{x}}^*, \mathbf{P}^*), \mathcal{N}(\mathbf{x}; \bar{\mathbf{x}}, \mathbf{P})\right)$$

From Proposition 2, the posterior pdf $p\left(\mathbf{x}_1|\mathbf{z}_1\right)$ can be approximated by a Gaussian pdf:

$$p\left(\mathbf{x}_1|\mathbf{z}_1\right) \approx \mathcal{N}(\mathbf{x}; \hat{\mathbf{x}}_1^*, \mathbf{P}_1^*) \tag{29}$$

Then, the Bayesian inference can proceed with the approximation (29).

Based on the discussion above, a suboptimal filter can be derived and its main steps are summarized in Table I.

TABLE I
A SUBOPTIMAL FILTER FOR THE SYSTEM WITH STATE CONSTRAINTS

S1: The initial state $\mathbf{x}_0 \sim \mathcal{N}(\hat{\mathbf{x}}_0, \mathbf{P}_0)$.
For $k = 0, 1, \cdots$
S2: Prediction
$p\left(\mathbf{x}_{k+1}|\mathbf{z}_k\right) \propto \mathcal{N}(\mathbf{x}_{k+1}; \hat{\mathbf{x}}_{k+1|k}, \mathbf{P}_{k+1|k})\mathbf{1}_{\{\alpha'\mathbf{x}_{k+1}\in[a,b]\}}$
where
$\hat{\mathbf{x}}_{k+1|k} = F_k\hat{\mathbf{x}}_k,\ \mathbf{P}_{k+1|k} = F_k\mathbf{P}_kF_k' + Q_k$
S3: Update
$p\left(\mathbf{x}_{k+1}|\mathbf{z}_{k+1}\right) \propto \mathcal{N}(\mathbf{x}_{k+1}; \hat{\mathbf{x}}_{k+1}, \mathbf{P}_{k+1})\mathbf{1}_{\{\alpha'\mathbf{x}_{k+1}\in[a,b]\}}$
where
$\hat{\mathbf{x}}_{k+1} = \hat{\mathbf{x}}_{k+1|k} + \mathbf{K}\left(\mathbf{z}_{k+1} - H_{k+1}\hat{\mathbf{x}}_{k+1|k}\right)$
$\mathbf{P}_{k+1} = \mathbf{P}_{k+1|k} - \mathbf{K}\mathbf{S}\mathbf{K}'$
and $\mathbf{S} = H_{k+1}\mathbf{P}_{k+1|k}H_{k+1}' + R_k,\ \mathbf{K} = \mathbf{P}_{k+1|k}H_{k+1}'\mathbf{S}^{-1}$
S4: Computing the mean $\hat{\mathbf{x}}_{k+1}^*$ and covariance $\mathbf{P}_{k+1}^*$ using Proposition 1;
S5: Approximate the posterior pdf by $\mathcal{N}\left(\mathbf{x}_{k+1}; \hat{\mathbf{x}}_{k+1}^*, \mathbf{P}_{k+1}^*\right)$
S6: Increase k and go to S2.

V. GROUND MOVING TARGET TRACKING WITH ROAD CONSTRAINTS

We give an illustrative example of the state estimation with constraints in this section.

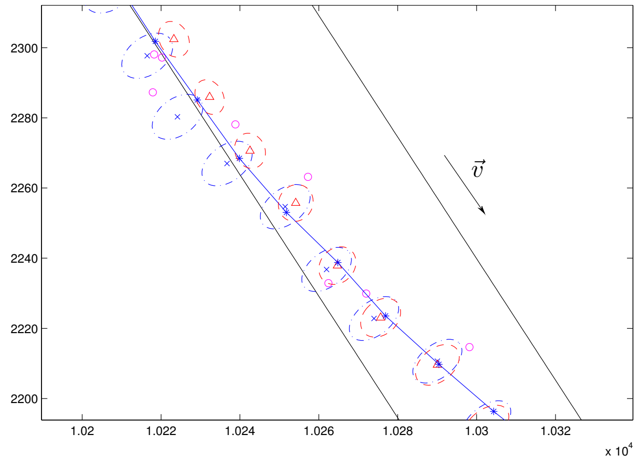

Fig. 3. The true trajectory and tracking with and without road constraints

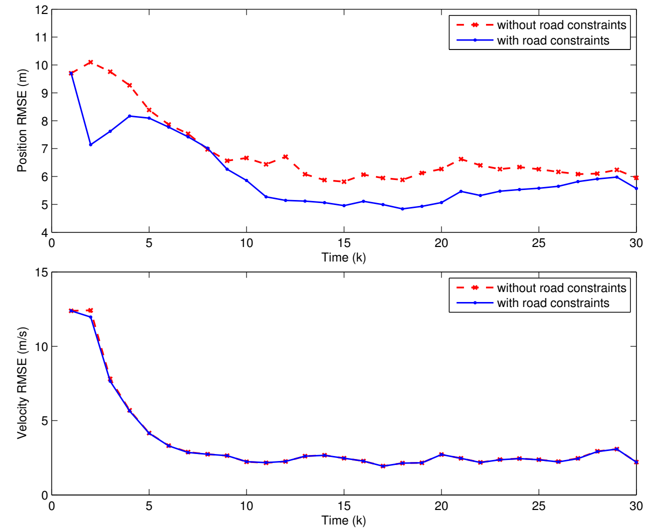

Fig. 4. RMS position and velocity errors comparison

Ground moving targets might be forced to stick to a road, e.g., on a bridge, which imposes hard constraints on the target position. In general, the road structure is represented by a number of segments linking a series of waypoints p and an associated width w. The ith segment is a rectangle defined by the three parameters $(\mathsf{p}_i, \mathsf{p}_{i+1}, \mathsf{w}_i)$. With a little algebraic manipulation, the parameters a, b and α can be easily obtained. In this scenario, $a = 10^4$, $b = 10040$ and $\alpha = [\sin(\pi/3), 0, \cos(\pi/3), 0]'$. Assume that the target moves on the ground with a nearly constant velocity and its motion model can be expressed by (24) and (25) with

$$F_k = \text{diag}[F_2, F_2],\ F_2 = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix},\ T = 1\text{s}$$
$$Q_k = \text{diag}[Q_2, Q_2],\ Q_2 = q\begin{bmatrix} T^3/3 & T^2/2 \\ T^2/2 & T \end{bmatrix},\ q = 1$$
$$H_k = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},\ R_k = \text{diag}[100\text{m}^2, 100\text{m}^2]$$

and the initial state $\mathbf{x}_0 = [10050\text{m},\ 10\text{m/s},\ 2625\text{m}, -17.3\text{m/s}]'$. We use the suboptimal filter developed in Section IV to estimate the target's state $\mathbf{x} = [x\ \dot{x}\ y\ \dot{y}]'$. Fig. 3 presents the tracking result in one run. In the figure, stars indicate the true trajectory of the target constrained in the road, circles (magenta) are the measurement sequence, triangles indicate the position estimate of the constrained tracker, dash (red) lines indicate the corresponding error ellipses, cross signs indicate the position estimate of the unconstrained tracker, and dash-dot (blue) lines indicate the corresponding 1-σ error ellipse. It can be seen that the constrained tracker results in much more accurate estimates and smaller error covariance than the unconstrained tracker. 500 Monte Carlo runs are conducted and the RMSEs for position and velocity are shown in Fig. 4. It is observed that this suboptimal filter incorporating the constraints information has better performance: its position accuracy is improved significantly by the position constraints information, while for the velocity, its improvement is minor in this scenario.

VI. Conclusions

This paper has considered the problem of estimating parameters or state variables constrained by a class of linear inequalities. For the constrained parameter estimation, we have presented two optimal sequential methods in the MMSE sense. The result is in an analytic form, different from most existing numerical methods.

To overcome the disadvantage that most existing models for the dynamic system with state constraints do not match the reality closely, we have proposed the target system in the form of density function and developed a suboptimal filter based on a reasonable approximation. An example of tracking a ground moving target has demonstrated the effectiveness of the filter.

Because of its analytic form, the proposed filter has less computation than the particle filter and moreover better performance than the conventional projection method. In the future, we will combine this filter with the variable structure multiple model approach to track ground moving targets and compare its performance with those of existing work.

Appendix

Proof: Let $z_c = \alpha'\mathbf{x}$ and $\mathcal{B} = \{z_c \in [a, b]\}$. In this case, the linear inequality constraint can be formulated as an additional noise-free measurement z_c constrained by bounds, that is,

$$p\left(\mathbf{x}|\mathbf{x} \in \mathcal{A}\right) \Leftrightarrow p\left(\mathbf{x}|\mathcal{B}\right)$$

Hence, the conditional expectation and covariance are

$$E\left(\mathbf{x}|\mathbf{x} \in \mathcal{A}\right) = E\left(\mathbf{x}|\mathcal{B}\right)$$
$$\text{Cov}\left(\mathbf{x}|\mathbf{x} \in \mathcal{A}\right) = \text{Cov}\left(\mathbf{x}|\mathcal{B}\right)$$

[16], [23] and [17] have studied the problem of the conditional expectation and covariance of a random variable with a scalar measurement lying in a specified interval. Apply

their results to the estimation problem with noise-free measurements constrained by bounds, and then we can obtain

$$E(\mathbf{x}|\mathcal{B}) = \bar{\mathbf{x}} + K\left[E(z_c|\mathcal{B}) - \alpha'\bar{\mathbf{x}}\right] \tag{30}$$

$$\mathrm{Cov}(\mathbf{x}|\mathcal{B}) = \mathbf{P} - K\mathrm{var}(z_c)K' + K\mathrm{var}(z_c|\mathcal{B})K' \tag{31}$$

where

$$K = \mathbf{P}\alpha\left(\alpha'\mathbf{P}\alpha\right)^{-1}$$
$$\mathrm{var}(z_c) = \alpha'\mathbf{P}\alpha$$
$$\mathrm{var}(z_c|\mathcal{B}) = E\left(z_c^2|\mathcal{B}\right) - \left[E(z_c|\mathcal{B})\right]^2$$

Furthermore, the conditional expectation of z_c is

$$\begin{aligned} m &\triangleq E(z_c|\mathcal{B}) \\ &= \frac{1}{c}\int_a^b z_c \mathcal{N}(z_c;\alpha'\bar{\mathbf{x}},\alpha'\mathbf{P}\alpha)\mathrm{d}z_c \\ &= \alpha'\bar{\mathbf{x}} - \frac{\alpha'\mathbf{P}\alpha}{c}\left[\mathcal{N}(b;\alpha'\bar{\mathbf{x}},\alpha'\mathbf{P}\alpha) - \mathcal{N}(a;\alpha'\bar{\mathbf{x}},\alpha'\mathbf{P}\alpha)\right] \end{aligned}$$

and the corresponding variance is

$$\begin{aligned} \Sigma &\triangleq \mathrm{var}(z_c|\mathcal{B}) \\ &= \frac{1}{c}\int_a^b z_c^2 \mathcal{N}(z_c;\alpha'\bar{\mathbf{x}},\alpha'\mathbf{P}\alpha)\mathrm{d}z_c - m^2 \\ &= -\frac{\alpha'\mathbf{P}\alpha}{c}\left[(b+\alpha'\bar{\mathbf{x}})\mathcal{N}(b;\alpha'\bar{\mathbf{x}},\alpha'\mathbf{P}\alpha)\right. \\ &\quad \left. -(a+\alpha'\bar{\mathbf{x}})\mathcal{N}(a;\alpha'\bar{\mathbf{x}},\alpha'\mathbf{P}\alpha)\right] + \alpha'\bar{\mathbf{x}}\bar{\mathbf{x}}'\alpha + \alpha'\mathbf{P}\alpha - m^2 \end{aligned}$$

where $c = \int_a^b \mathcal{N}(z_c;\alpha'\bar{\mathbf{x}},\alpha'\mathbf{P}\alpha)\mathrm{d}z_c$. This completes the proof. ■

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